## CURVES AND SURFACES WITH CONSTANT CURVAURE IN EUCLIDEAN SPACE

### I. Curves with constant cuvaure in Euclidean space

At each point there are often lines with a constant bend. Let's consider lines on the plane and in space that have this property.

1) ***Circle***. Let's write a circle parametrically and calculate its bend at any point M(t).

$$x = a\cdot Cost,\ y = a\cdot Sint,\ t\in[0, 2\pi].$$

We'll find out. x'= - $a$Sint, y' = $a$Cost, and x'' = -$a$Cost, y''= -$a$Sint.

Then for flat lines, we find the curve expression at any point M(t):

$$k = \left|\frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{\sqrt{((x'')^2+(y')^2)^3}}\right|,\ k = \left|\frac{\begin{vmatrix} -aS\mathrm{int} & aCost \\ -aCost & -aS\mathrm{int} \end{vmatrix}}{\sqrt{((-aS\mathrm{int})^2+(aCost)^2)^3}}\right| = \frac{a^2}{a^3} = \frac{1}{a} = \text{Const., and radius } \rho = a.$$

2) ***Screw (screw) line***.

*Definition 1*. If a point located in a plane perpendicular to a straight line in space moves along a straight line parallel to a given straight line with a uniform rotation from this straight line, then a line formed from its traces is called a screw (screw) line. The screw line can be defined by the regular Line L given as follows:

L: **r**(t) = ($a$Cost, $a$Sint, bt), $t\in(-\infty,\infty)$ (*), here $a$ and b are positive numbers. (*) each point of a parametrically expressed line corresponds to a circular cylindrical surface defined by the equation $x^2+y^2=a^2$ (fig.1). (*) calculate the bending of the screw line at any point M (t) given parametrically. Since **r'**= - $a$Sint·**i** + $a$Cost·**j** + b·**k**, **r''** = - $a$Cost·**i** - $a$Sint·**j** is a vector product

$$[\mathbf{r}', \mathbf{r}''] = \begin{vmatrix} -aS\mathrm{int} & -a\cos t & i \\ aCost & -aS\mathrm{int} & j \\ b & 0 & k \end{vmatrix} = ab\mathrm{Sint}\cdot\mathbf{i} - ab\mathrm{Cost}\cdot\mathbf{j} + a^2\cdot\mathbf{k} \text{ and } |[\mathbf{r}', \mathbf{r}'']| = a\cdot\sqrt{a^2+b^2}.$$

Then calculate the bending by the expression $k = \frac{|[r', r'']|}{|r'|^3}$ the bending of the screw at each point of the line: $k = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = = \frac{a}{a^2+b^2}$ = Const $k = \frac{|[r', r'']|}{|r'|^3}$, and the radius of the bend $\rho = \frac{a^2+b^2}{a}$

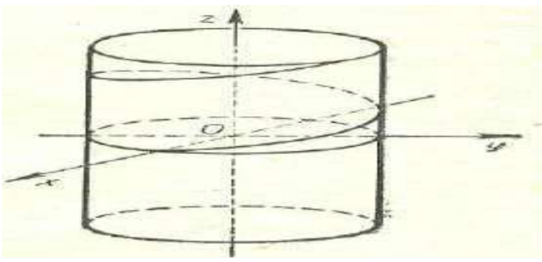

fig.1.

### II. Examples of surfaces of constant total and average curvature

*Definition 2*. A surface **F** is called a surface of constant total (respectively, average) curvature if at all points of this surface K=const, H=const.

*Examples*.

1) a plane or part of It. .

2) Cylindrical and conical surfaces.

*Definition*. The surface in which the average curvature is zero, are called minimal.

A straight helicoid is a minimal surface.

**a)** *Rotation surfaces*. It is known that rotation surfaces are a surface obtained from rotation from a straight line located in the plane of a given line. To write its equation, consider a rectangular coordinate system. let the z increment be the rotation increment. Let P be the plane of a smooth line $\gamma$. Let's introduce a rectangular Ouz coordinate system in this plane, where u = P$\cap$(Oxy). let's assume that the $\gamma$ line is defined in the Ouz coordinate system by the equation z = f(u). let's $\varphi$ denote the angle between the x and the u. When any point M(x, y, z) rotates around a circle from z, the angle $\varphi$ changes at [0, 2 ] intervals. Then the parametric equation of the rotation surface **F** obtained in the Oxyz coordinate system will be as follows:

$$x = u\cdot Cos\varphi,\ y = u\cdot Sin\varphi,\ z = f(u),\ \text{or}$$

$$\mathbf{r} = (u\cdot Cos\varphi)\cdot i + (u\cdot Sin\varphi)\cdot j + f(u)\cdot k,$$

here the rank $\begin{pmatrix} x_u & y_u & z_u \\ x_\varphi & y_\varphi & z_\varphi \end{pmatrix} = \begin{pmatrix} Cos\varphi & Sin\varphi & f'(u) \\ -u\cdot Sin\varphi & u\cdot Cos\varphi & 0 \end{pmatrix} = 2$ here the rank (u, $\varphi$)$\in$G, u$\neq$0, must be executed at the point.

On this page, the coordinate lines u, $\varphi$=const are meridians, and u= cosnt, $\varphi$ it's called parallels. We find: $\mathbf{r}_u = (Cos\varphi, Sin\varphi, f'(u))$, $r_\varphi = (-u\cdot Sin\varphi, u\cdot Cos\varphi, 0)$, $\mathbf{r}_{uu} = (0, 0, f''(u))$,

$r_{u\varphi} = (-Sin\varphi, Cos\varphi, 0)$, $r_{\varphi\varphi}(-u\cdot Cos\varphi, -u\cdot Sin\varphi, 0)$, then the first and coefficients of second quadratic forms: $\gamma_{11} = 1 + (f'(u))^2$, $\gamma_{12} = 0$, $\gamma_{22} = u^2$, $b_{11} = \dfrac{f''(u)}{\sqrt{1+(f'(u))^2}}$, $b_{12} = 0$,

$$b_{22} = \frac{u\cdot f'(u)}{\sqrt{1+(f'(u))}}.$$

$\gamma_{12} = 0$, $b_{12} = 0$ since lines-meridians and parallels – will be bend lines.

The full Bend will be

$$K = \frac{f'(u)\cdot f''(u)}{u\cdot(1+(f'(u))^2)^2}. \qquad (**)$$

3) ***Sphere***. In the Ouz coordinate plane, the equation of the meridian of the sphere is $z^2 + u^2 = a^2$, where $a$ - is the radius of the sphere. let z = $\sqrt{a^2-u^2}$. Hence, ff(u) = $\sqrt{a^2-u^2}$.

Find: $f'(u) = -\dfrac{u}{\sqrt{a^2-u^2}}$, $f''(u) = -\dfrac{a^2}{(a^2-u^2)\cdot\sqrt{a^2-u^2}}$.

(**) the complete bending of the sphere by equality K= $\dfrac{1}{a^2}$ = const, that is, the sphere is a constant non-negative malleable surface.

*4) **Pseudosphere***. The surface formed by the rotation of the innkeeper's own growth is called the pseudosphere (fig.2).

Parameter equation of the tractor: z = $a\cdot(\ln tg\dfrac{t}{2} + Cost)$, $u = a\cdot S\mathrm{int}$, $a = const > 0$.

Then, $z'_t = \dfrac{a\cdot Cos^2 t}{S\mathrm{int}}$, $u'_t = a\cdot Cost$. $f'(u) = \dfrac{z'_t}{u'_t} = Ctgt$, $t \neq 0$. Here (*) we find by equality:

K = - $\dfrac{1}{a}$, that is, the pseudosphere will be a constant negative malleable surface.

*Pseudosphere*

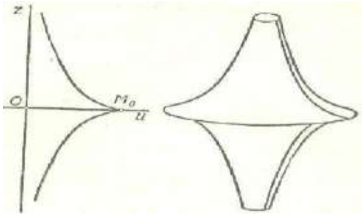

fig.2.

*Cylindrical* and *conical surfaces* of rotation are also examples of surfaces with constant bending, since all points of these surfaces are parabolic, that is, at each point K = 0.

5) ***Helicoid*** . Calculate the total and average curvature of a straight helicoid.

Parametric equations of the helicoid: **F:** x=uCosv, y=uSinv, z=bv, b>0.

Then, $r_u = (Cosv; Sinv; 0), r_v = (-uSinv; uCosv; b) \Rightarrow \gamma_{11} = 1, \gamma_{12} = 0, \gamma_{22} = u^2 + b^2$,

$r_{uu} = (0,;0;0), r_{uv} = (-Sinv; Cosv, 0), r_{vv} = (-uCosv; -uSinv; 0) \Rightarrow$

$$b_{11} = 0, b_{12} = -\frac{b}{\sqrt{u_2 + b_2}}, b_{22} = 0 \Rightarrow K = -\frac{b^2}{(u^2 + b^2)}, H = 0.$$

<u>*Definition 3*</u>. The surface in which the average curvature is zero, are called minimal.

A straight helicoid is a minimal surface.

Literature


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